PSEUDO-MACROCRACK WITH A "QUASI-PLASTIC" CRACK-OPENING—LOAD LAW

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1. Introduction. In the present study, we examine a pseudo-macrocrack. This is a crack which is located in a composite or nonuniform body and has edges that are confined by unfractured elements of the structure. The classical example of a crack of this type is a macrocrack in a composite with a brittle ceramic matrix and intact ductile fibers which constrain its edges and prevent its opening [1, 2]. The most general formulation was used in [3] to study a single pseudo-macrocrack in an elastic, linearly anisotropic body under plane strain conditions. Here, the relationship between the forces transmitted from one edge to the other σ_{ni} and the opening of the edges $w = [v_y(x)]$ was assumed to be linear:

$$\sigma_y(x) = kw(x). \tag{1.1}$$

The following constraint parameter was introduced

$$\lambda = 2k\frac{1-\nu_{yz}\nu_{zy}}{E_y}\,\mathrm{Re}\left(i\frac{\mu_1+\mu_2}{\mu_1\mu_2}\right), \tag{1.2}$$

this parameter having the dimension of inverse length (λ^{-1} is of the same order of magnitude as the characteristic dimension of the structure of the material). Also, it was shown from the condition

$$\lambda l \gg 1 \tag{1.3}$$

that the critical load p_* for advance of the pseudo-macrocrack is independent of its length 2ℓ [3]:

$$p_* = \sqrt{4\gamma_p k}. \tag{1.4}$$

It was further assumed that propagation of the tip of the pseudo-macrocrack occurs in a quasi-brittle manner. Above, $2\gamma p$ is the unit dissipation of energy during its advance. It was also shown in [3] that condition (1.3) (which is realized for pseudo-macrocrack lengths greater than several periods of the structure) makes it possible to to efficiently construct a solution in the form of an asymptotic series in inverse powers of the dimensionless parameter $\lambda\ell$.

Nevertheless, it is evident that the linear relational law $\sigma - w$ (1.1) examined in [3] does not even come close to fully covering the wide variety of practical situations or engineering requirements that are encountered. For example, it follows from the analysis made in [4, 5] that in the separation of intact fibers from the matrix or slip behind the front of a growing crack, the constitutive equation corresponding to (1.1) has the form $\sigma^\pm \sim \sqrt{w}$. More complex situations are also possible. In connection with this, below we construct the resolvent equation of the plane problem of the anisotropic theory of elasticity for a single pseudo-macrocrack with a nonlinear "quasi-plastic" law assigned for its edges

$$\sigma_y^\pm = p\,(w\,(x)) \tag{1.5}$$

and we obtain its solution in specific cases. We find the limit loads for a body with a pseudo-macrocrack for certain special forms of the function p(w). It is assumed that the function p(w) is determined by the structure of the material and the mode of fracture and is independent of the size or shape of the pseudo-macrocrack. It must either be known a priori or be found from independent experiments.

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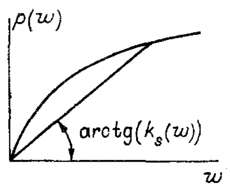

Fig. 1

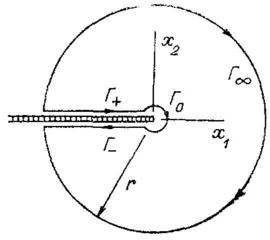

Fig. 2

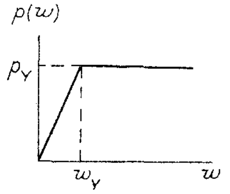

Fig. 3

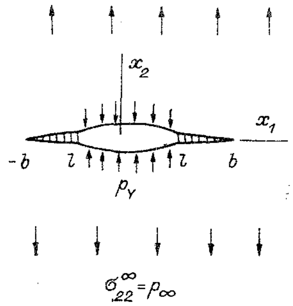

Fig. 4

For the sake of definiteness, we will examine only active loading. Thus, consideration of the processes which occur during unload is unnecessary. The only restriction we impose on the function p(w) is that it not decrease. This ensures that the solution of the problem will be unique. Also, under the given conditions, the "response" of the elastic body to an introduced pseudo-macrocrack with interaction law (1.5) (including the limit load) is independent of the actual physical nature of the interaction of the edges: the function p(w) can describe purely nonlinearly elastic interaction of the edges, purely plastic behavior of the structural elements joining the edges of the crack, the interaction of ideally elastic components in the presence of friction, and other phenomena.

2. Formulation of the Problem and Asymptotic Solution. We will examine a pseudo-macrocrack occupying the region $|x_1| < l$, $x_2 = 0$, $|x_3| < \infty$ of an infinite anisotropic space. We will assume that the normal axis x_3 is a plane of elastic symmetry. In this case, the two-dimensional problem decomposes into two distinct problems: longitudinal shear and planar deformation. At infinity, we assign a uniform field of tensile stresses

$$\sigma_{22}^{\infty} = p_{\infty}, \quad \sigma_{12}^{\infty} = \sigma_{11}^{\infty} = 0. \tag{2.1}$$

With unequal roots μ_r (r = 1, 2) for the characteristic equation, the stresses σ_{11}, σ_{22}, σ_{12} and displacements u, v are expressed through the two complex potentials of S. G. Lekhnitskii [6]:

$$\begin{gathered}\sigma_{11} = 2\mathrm{Re}\left(\mu_1^2\Phi_1(z_1) + \mu_2^2\Phi_2(z_2)\right), \quad \sigma_{22} = 2\mathrm{Re}\left(\Phi_1(z_1) + \Phi_2(z_2)\right), \\ \sigma_{12} = -2\mathrm{Re}\left(\mu_1\Phi_1(z_1) + \mu_2\Phi_2(z_2)\right), \quad u = 2\mathrm{Re}\left(p_1\varphi_1(z_1) + p_2\varphi_2(z_2)\right), \\ v = 2\mathrm{Re}\left(q_1\varphi_1(z_1) + q_2\varphi_2(z_2)\right).\end{gathered} \tag{2.2}$$

Here, $z_r = x_1 + \mu_r x_2$; p_r, q_r are complex parameters [6]; $\Phi_r(z_r) \equiv \varphi_r'(z_r)$ $(r = 1, 2)$. Conditions (1.5) are assigned on the edges of the pseudo-macrocrack. Meanwhile, by virtue of symmetry

$$\sigma_{22}^{\pm}(x_1) = p(w(x_1)), \quad \sigma_{12}^{\pm}(x_1) = 0, \quad |x_1| < l \tag{2.3}$$

$(w(x) = v^+(x) - v^-(x))$. The following relations are satisfied on the remaining part of the x_1 axis

$$\sigma_{12}^{\pm}(x_1) = 0, \quad v(x_1) = 0, \quad |x_1| > l. \tag{2.4}$$

It is convenient to look for the solution of the given problem in the form of Cauchy integrals, as was done in [3]:

$$\varphi_r(z_r) = \Gamma_r z_r + \frac{\mu_t}{2\pi}\int_{-l}^{+l}\frac{q(x)}{\mu_t-\mu_r}\frac{dx}{x-z_r} \quad (2.5)$$

(Γ_r are constants which determine the field (2.1) at infinity, while q(x) is a continuous real-valued function which must be determined). In Eq. (2.5), the subscripts r and t alternately take values of 1 and 2: if r = 1, then t = 2, and conversely. The unknown real function q(x) is connected with the opening of the pseudo-macrocrack w(x) ≡ [v(x)] by the relation [3]

$$w(x) = \alpha q(x), \quad \alpha = 2\frac{1-\nu_{yz}^2}{E_y}\operatorname{Re}\left(i\frac{\mu_1+\mu_2}{\mu_1\mu_2}\right). \quad (2.6)$$

The condition of continuity of the displacements reduces to

$$q(l) = q(-l) = 0. \quad (2.7)$$

In this case (see [3]), Eqs. (2.2), (2.5) identically satisfy all boundary conditions (2.1), (2.3-2.4) except for the first condition of (2.3). The distribution of the stresses $\sigma_{22}(x)$ on the x_1 axis is given by the formula

$$\sigma_{22}^+(x) = \sigma_{22}^-(x) = p_\infty + \frac{1}{\pi}\int_{-l}^{+l}\frac{q'(\xi)}{\xi-x}d\xi, \quad (2.8)$$

where the integral is understood in the sense of the principal Cauchy value for the interior points of the interval $[-\ell, \ell]$. Satisfying the first condition in (2.3) by means of (2.6), (2.8), we obtain a nonlinear singular integrodifferential equation

$$p(\alpha q(x)) - \frac{1}{\pi}\int_{-l}^{+l}\frac{q'(\xi)}{\xi-x}d\xi = p_\infty \quad (2.9)$$

for the unknown function q(x). This equation can be reduced to a nonlinear integral equation with a logarithmic kernel and can be solved numerically. Here, we will examine the conditions under which Eqs. (2.7), (2.9) can be solved asymptotically.

Let us evaluate the function q(x) at $x \to \pm\ell$. If we allow for (2.7) and make use of the properties of Cauchy integrals near the ends of the line of integration [7] in the case of Eq. (2.9), we obtain the usual root relation for crack-opening w(x)

$$q(x) \simeq 2N\sqrt{l-|x|} \quad (2.10)$$

for any function p(w) which is finite in zero ($w \to 0$). The constant N is connected with the stress-intensity factor K_I at the tip of the pseudo-macrocrack by the relation

$$K_I = N\sqrt{2\pi}. \quad (2.11)$$

Thus, the asymptote of the stress-strain state around the tip of a pseudo-macrocrack coincides with the usual asymptote. We introduce the quantity $w_0 \equiv \max\{w(x_1)\}$, $x_1 \in [-l, l]$ being the maximum opening of the pseudo-macrocrack at the given level of applied loads (in the case of boundary conditions (2.1), this maximum will obviously be reached at the center of the pseudo-macrocrack). Then we can use the secant modulus p(w) (Fig. 1) $k_p(w) = p(w)/w$ to find a condition which is equivalent to the inequalities (1.3) of the linear theory and determines a small parameter that can be used to expand the solution. This condition can be represented in the form

$$\lambda_p l \gg 1, \ \lambda_p \equiv \alpha k_p(w_0). \quad (2.12)$$

Accordingly, we can take $(\lambda_p\ell)^{-1}$ as the small parameter.

If inequality (2.12) is satisfied, then the solution of Eq. (2.9), asymptotically correct for points of interval $[-\ell, \ell]$ which are not too close to its ends $(|x| \leqslant l - \lambda_p^{-1})$, takes the form

$$q(x) = w_\infty \alpha^{-1}(1 + O(\lambda_p l)^{-1})), \tag{2.13}$$

where by definition we have $w_\infty \equiv w(p_\infty)$, where w(p) is the inverse of p(w). The physical value of w_∞ is the divergence of the edges of the pseudo-macrocrack at a sufficient distance from its ends.

Let us calculate the stress-intensity factor K_I at the tip of the pseudo-macrocrack. As in the case of a pseudo-macrocrack with linear constraints, the factor will be independent of the length of the crack if conditions (2.12) are satisfied. Accordingly, the limit loads will also be independent of the crack's length. Having this in mind, we will examine a semi-infinite pseudo-macrocrack under the same loads (2.1) as above. We calculate the Eshelby–Cherepanov–Rice integral

$$J = \int_\Gamma \left(\frac{1}{2} \sigma_{ij}\varepsilon_{ij}\, dy - \sigma_{ij} n_j \frac{\partial v_i}{\partial x}\, d\Gamma \right), \tag{2.14}$$

taken over the closed contour $\Gamma = \Gamma_0 \cup \Gamma_+ \cup \Gamma_- \cup \Gamma_\infty$, shown in Fig. 2 and, thus, identically equal to zero. Having separated it into the sum of integrals, we obtain

$$J_0 = -J_\infty - J_+ - J_-. \tag{2.15}$$

Constricting the contour Γ_0 toward the tip of the pseudo-macrocrack and allowing for the path of the integral, we find that

$$J_0 = -(\alpha/4)(K_I)^2. \tag{2.16}$$

The integral J_∞ over the infinitely distant contour Γ_∞ approaches zero, since the stress and strain fields are composed of a uniform external field and fields connected with the pseudo-macrocrack. The latter decay in accordance with the law $\sim r^{-1}$ at distances from the tip $r \gg \lambda_p^{-1}$, so that $J_\infty \sim r^{-1}$. The sum of the integrals is easily represented in the form

$$J_+ + J_- = \int_{-r}^{0} \sigma_{22}^+(x) \frac{\partial v^+}{\partial x}\, dx + \int_{0}^{-r} \sigma_{22}^-(x) \frac{\partial v^-}{\partial x}\, dx.$$

Transforming this expression, we find $J_+ + J_- = \int_0^{-r} \sigma_{22}^{\pm}(x)\, dw(x)$. Making a substitution of variables here with allowance for condition (2.3) and the fact that $w(p) \to w_\infty$ at $r \to \infty$, we obtain

$$J_+ + J_- = \int_0^{w_\infty} p(w)\, dw. \tag{2.17}$$

Combining (2.15-2.17), we finally obtain

$$K_I = \sqrt{\frac{4}{\alpha} \int_0^{w_\infty} p(w)\, dw}. \tag{2.18}$$

With linear constraint law (1.1) $p(w) = kw$, Eq. (2.18) becomes $K_I = p_\infty \sqrt{2/\lambda}$ (λ is determined from (1.2)). This quantity (K_I) was calculated in [3] by a different method.

We derive the condition for the advance of a pseudo-macrocrack with "plastic" constraints from the Irwin criterion. Here, we equate the calculated value of K_I (2.18) to its critical value K_{I_c}:

$$K_I = K_{Ic} \equiv \sqrt{\frac{4\gamma_p E_y}{\varkappa(1-\nu_{yz}\nu_{zy})}}, \quad \varkappa = \mathrm{Re}\left(i\frac{\mu_1+\mu_2}{\mu_1\mu_2}\right). \tag{2.19}$$

Inserting (2.18) into (2.19), we find the final condition for advance of the pseudo-macrocrack in the form

$$\int_0^{w_\infty^*} p(w)\,dw = 2\gamma_p, \tag{2.20}$$

where w^*_∞ should be taken as the value of the function $w(p_\infty)$ at $p_\infty = p^*$. With the use of the inverse function w(p), this same formula can be represented as $w_\infty^* p^* - \int_0^{p^*} w(p)\,dp = 2\gamma_p$.

It should be noted that, with linear law p(w) = kw, after calculation of the integral criterion (2.20) reduces to Eq. (1.4) — which was obtained in [3] by a different method.

It is fully valid to use condition (2.19) as the fracture criterion, since in accordance with (2.10) a purely linearly elastic stress state is realized in the immediate vicinity of the crack tip, and all possible dissipative processes occur at a certain distance ($\sim\chi_p^1$) from it. Thus, the dynamic equilibrium of the tip of a pseudo-macrocrack is determined exclusively by theoretical stress-intensity factor K_I (2.18) and the fracture work of the material $2\gamma_p$. The quantity $2\gamma_p$ does not include friction losses, fiber pull-out, or other forms of energy dissipation not associated directly with the processes occurring at the tip.

Thus, as noted above, the critical load for the advance of pseudo-macrocrack depends only on the form of the function p(w) — the physical meaning of which may be completely arbitrary.

As an example, we will examine a pseudo-macrocrack with a logarithmic edge-constraint law:

$$p(w) = p_0 \ln\left(1 + \frac{k}{p_0} w\right) \tag{2.21}$$

(p_0 and k are fixed constants). Inserting (2.21) into (2.20), we obtain an equation connecting the critical load p^* with the fracture work $2\gamma_p$ and the constraint parameters p_0, k:

$$(p^*/p_0 - 1)\exp(p^*/p_0) + 1 = 2\gamma_p k/p_0^2, \tag{2.22}$$

from which at $p^* \ll p_0$ we find $p^* = \sqrt{4\gamma_p k}$, while at $p^* \gg p_0$ we have $p^* = p_0 \ln(2\gamma_p k/p_0^2)$. After the solution of the problem, it is necessary to check for satisfaction of inequality (2.12). In the case of law (2.21), this condition reduces to the form

$$klp^*/p_0 \gg E_y(\exp(p^*/p_0) - 1). \tag{2.23}$$

If the value of p^* found from Eq. (2.22) after substitution of this equality into (2.23) does not violate the latter, then the above asymptotic approach will give accurate results throughout the possible range of external loads p_∞, from zero to p^*. The example we have examined shows that, generally speaking, for a pseudo-macrocrack of length 2ℓ with a nonlinear edge-constraint law (1.5), there may be a region of external loads p within which either the above asymptotic approach is invalid or violation of inequality (2.12) leads to deviations from Eqs. (2.13), (2.18) (see Part 3).

Without detailing the steps taken, let us present the limit load p^* and criterion of applicability of the asymptotic formulas in the case of a power edge-interaction law: $p(w) = kw^{1/s}$ ($s > 1$). In accordance with (2.19), the limit load p^* is expressed by the formula $p^* = (2\gamma_p(1 + 1/s)k^s)^{1/(s+1)}$, while the condition of its applicability is $lk^s \gg E_y(p^*)^{s-1}$.

3. Pseudo-Macrocrack with Elastic—Ideally Plastic and Plastic-Rigid Constraints. The case of a pseudo-macrocrack of length 2b with elastic—ideally plastic constraints de-

termined by the formula

$$p(w)=\begin{cases}kw, & \text{if} \quad w<w_Y,\\ p_Y\equiv kw_Y, & \text{if} \quad w>w_Y\end{cases} \tag{3.1}$$

(Fig. 3) is deserving of special attention. We will assume that the condition

$$kb \gg E_y \tag{3.2}$$

is satisfied. If the external load $\sigma_y^\infty = p_\infty$ is less than the limit load p_Y, then the solution of this problem trivially follows from the solution of the problem of a pseudo-macrocrack with linear constraints. If $p_\infty > p_Y$, then the solution formally coincides with the solution of the problem of a macrocrack of length 2ℓ ($\ell < b$) initiated from the ends of a pseudo-macrocrack of the size $b - \ell$ (Fig. 4) [8]. In accordance with (3.1), the following conditions are satisfied on the sections $|x| < \ell$

$$\sigma_{22}^{\pm}(x) = p_Y, \quad \sigma_{12}^{\pm}(x) = 0, \tag{3.3}$$

while on the sections $\ell < |x| < b$

$$\sigma_{22}^{\pm}(x) = kw, \quad \sigma_{12}^{\pm}(x) = 0. \tag{3.4}$$

The solution of problem (3.3-3.4) is represented by Cauchy integrals, analogous to (2.5). Meanwhile, as in [8], a singular integrodifferential equation can be obtained for the unknown real density of the potentials q(x) ($q(\pm b) = 0$):

$$p_\infty + \frac{1}{\pi}\int_{-b}^{+b}\frac{q'(\xi)}{\xi - x}\,d\xi = \begin{cases}p_Y, & |x|<l,\\ \lambda q(x), & l<|x|<b.\end{cases}$$

Here, as in [3, 8], λ is determined by Eq. (1.2). The unknown transition point ℓ is found from the condition $\lambda q(\ell) = p_Y$. Meanwhile, $q(x) > p_Y\lambda^{-1}$, on the central section, while at $l < |x| < b$ the condition $q(x) < p_Y\lambda^{-1}$. must be satisfied. It should be noted that that the value of the extreme sections of $b - \ell$ will be comparable to the length of the pseudo-macrocrack ($b - \ell \sim b$) only when the following condition is satisfied

$$0 < \frac{p_\infty - p_Y}{p_Y} \ll \frac{1}{\sqrt{\lambda b}}. \tag{3.5}$$

Since inequality (3.2) is equivalent to $\lambda b \gg 1$, condition (3.5) establishes only a relatively narrow range of external loads for which $b - \ell \sim b$ is possible. If the external load p_∞ does not satisfy the last inequality in (3.5), then the end regions will be small compared to the length of the pseudo-macrocrack: $b - \ell \ll b$.

Let us examine this situation in greater detail, not only because of its importance, but because it can be solved analytically. In the given case, there are two asymptotes near the ends of the pseudo-macrocrack: a near asymptote, at distances $r \ll \lambda^{-1}$, with the stress-intensity factor K_I; a far asymptote, at distances r such that $b - l \ll r \ll l$, $r \gg \lambda^{-1}$, with the stress-intensity factor

$$K_I^\infty = (p_\infty - p_Y)\sqrt{\pi b}. \tag{3.6}$$

Calculating the Eshelby—Cherepanov—Rice integral (2.14) over the contours depicted in Fig. 2 (with the contour Γ_0 extending to within the range of the near asymptote and the contour Γ_∞ extending to within the range of the far asymptote), we find the relationship between K_I and K_I^∞:

$$\frac{1-\nu_{yz}\nu_{zy}}{2E_y}\varkappa\left(K_I^\infty\right)^2 = \frac{1-\nu_{yz}\nu_{zy}}{2E_y}\varkappa\left(K_I\right)^2 + k\frac{w^2(l)}{2} - p_Y w(l).$$

Since $w(\ell) = w_Y = p_Y/k$ at the point $x = \ell$, the last relation takes the simple form

$$(K_I^\infty)^2 = (K_I)^2 - 2p_Y^2/\lambda. \tag{3.7}$$

Finally, considering (3.6), we obtain the final expression

$$K_I = \sqrt{(p_\infty - p_Y)^2 \pi b + 2p_Y^2/\lambda}. \tag{3.8}$$

Generally speaking, it is not always possible to ignore the last term in the radicand – even with allowance for $\lambda b \gg 1$ – since the level of the external load may turn out to be such that the first term in the radicand will be comparable to the second. This occurs, for example, if $p_\infty \sim p_Y(1 + \sqrt{2/\pi\lambda b})$.

The critical stresses p_* are determined by equating the calculated value of K_I (3.8) to its critical value (2.19):

$$p_* = p_Y + \sqrt{\frac{4\gamma_p E_y}{\varkappa(1 - \nu_{yz}\nu_{zy})\pi b} - \frac{2p_Y^2}{\pi\lambda b}}. \tag{3.9}$$

Expression (3.9) for p_* is valid only when $p_Y < \sqrt{4\gamma_p k}$. Otherwise, Eq. (1.4) is valid.

A pseudo-macrocrack with plastic-rigid constraints is the limiting case of the example examined above. The function p(w) (3.1) is taken in the form

$$p(w) = \begin{cases} 0, & \text{if} \quad w < w_Y, \\ p_Y, & \text{if} \quad w > w_Y. \end{cases}$$

Here, if external load p_∞ is less than p_Y, the pseudo-macrocrack does not disturb the uniform external field surrounding it. At $p_\infty > p_y$, the field of pseudo-macrocrack coincides with the field of an ordinary macrocrack subjected to action of uniform loads $\sigma_y = p_\infty$ at infinity and normal forces $\sigma^\pm_y = p_y$ on its edges. Stress-intensity factor is given by expression $K_I = (p_\infty - p_Y)\sqrt{\pi b}$, which leads to critical loads $p_* = p_Y + \sqrt{4\gamma_p E_y/[\varkappa(1 - \nu_{yz}\nu_{zy})\pi b]}$.

LITERATURE CITED

1. A. G. Evans, "The mechanical performance of fibre-reinforced ceramic matrix composites," Proc. 9th RISQ International Symposium on Metallurgy and Materials Sciences. "Mechanical Physical Behavior of Metallic and Ceramic Composites," Eds. by S. I. Andersen, H. Lilholt, and O. B. Pedersen, Roskilde (1988).
2. V. I. Kaz'min, S. T. Mileiko, and V. V. Tvardovskii, "Fracture of a model composite with a ceramic matrix," Mekh. Kompozit. Mater., No. 2 (1988).
3. V. V. Tvardovskii, "Toward a theory of pseudo-macrocracks in an anisotropic body. Part I. Single pseudo-macrocrack," Izv. Akad. Nauk SSSR Mekh. Tverd. Tela, No. 2 (1991).
4. D. B. Marshall, B. N. Cox, and A. G. Evans, "The mechanism of matrix cracking in brittle-matrix fibre composites," Acta Metall., 33, No. 11 (1985).
5. L. N. McCartney, "Mechanics of matrix cracking in brittle-matrix fibre-reinforced composites," Proc. R. Soc. Lond., A409, 329 (1987).
6. S. G. Lekhnitskii, Theory of Elasticity of an Anisotropic Body [in Russian], Nauka, Moscow (1977).
7. N. I. Muskhelishvili, Singular Integral Equations [in Russian], Gostekhizdat, Moscow (1946).
8. V. V. Tvardovskii, "Toward a theory of pseudo-macrocracks in an anisotropic body. Part II. Interaction of macrocracks and pseudo-macrocracks," Izv. Akad. Nauk SSSR Mekh. Tverd. Tela, No. 3 (1991).